

On steepest descent curves for quasi convex families in \mathbb{R}^n

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Abstract. A connected, linearly ordered path $\gamma \subset \mathbb{R}^n$ satisfying

$$x_1, x_2, x_3 \in \gamma, \text{ and } x_1 \prec x_2 \prec x_3 \implies |x_2 - x_1| \leq |x_3 - x_1|$$

is shown to be a rectifiable curve; a priori bounds for its length are given; moreover, these paths are generalized steepest descent curves of suitable quasi convex functions. Properties of quasi convex families are considered; special curves related to quasi convex families are defined and studied; they are generalizations of steepest descent curves for quasi convex functions and satisfy the previous property. Existence, uniqueness, stability results and length's bounds are proved for them.

Résumé. Nous démontrons que les chemins $\gamma \subset \mathbb{R}^n$ qui sont connectés et ordonnés, avec la propriété de monotonie

$$x_1, x_2, x_3 \in \gamma, \text{ et } x_1 \prec x_2 \prec x_3 \implies |x_2 - x_1| \leq |x_3 - x_1|$$

sont des courbes. Des limitations pour leur longueur sont prouvées. Ces chemins sont des généralisations de courbes de la plus grande pente pour appropriées fonctions quasiconvexes. Propriétés des familles quasiconvexes et courbes liées avec elles sont étudiées. Nous démontrons l'existence, l'unicité, la dépendance continue de ces courbes avec des limitations pour leur longueur.

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1 Introduction

Given a smooth, real valued function in a domain $A \subset \mathbb{R}^n$, $f : A \rightarrow \mathbb{R}$, with gradient $Df \neq 0$, steepest descent curves are solutions to

$$\dot{x}(t) = \phi(Df(x(t)))Df(x(t)), \quad \phi > 0 \tag{1}$$

(sometimes the condition $\phi < 0$ is preferred). The steepest descent curves satisfy the geometrical fact to be orthogonal to the level sets $\{x : f(x) = \text{const}\}$. This fact shows that the trajectories of the steepest descent curves depend on the level sets of f only.

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The definition of steepest descent curves has been extended to more general functions and spaces in [4], [8], [9], [10], [2], [16].

Here generalized steepest descent curves will be defined and studied for quasi convex functions without smoothness assumptions; let us refer to [5] for properties of these functions. In particular they will be studied for lower semi continuous bounded functions f with compact convex sub level sets:

$$\Omega_\tau = \{x : f(x) \leq \tau\} \quad \inf f \leq \tau \leq \sup f.$$

Let $\{\Omega_\tau\}$ be a nested family of bounded compact convex sets, sub level sets of a bounded, lower semi continuous, quasi convex function; this family, according to [12], will be called a **quasi convex family**. Let us assume here, for simplicity, that $\text{Int}(\Omega_\tau)$, the set of its interior points, is not empty. In the work it can happen that $\text{Int}(Q) = \emptyset$ for some Q in the family. The condition for a curve $x(\cdot)$ to be a steepest descent curve for a quasi convex family will be that,

$$x(\tau) \in \partial\Omega_\tau \quad \forall \tau \quad \text{and} \quad \dot{x}(\tau) \in N_{\Omega_\tau} \quad \text{a.e.}, \quad (2)$$

where N_{Ω_τ} is the normal cone to Ω_τ at $x(\tau)$; i.e. $x(\cdot)$ is a time-dependent trajectory of a differential inclusion, see e.g. [1, §4.4]. Properties of steepest descent curves for quasi convex families were observed in [15]: rectifiable curves were studied that are steepest descent curves for some quasi convex family $\{\Omega_\tau\}$ and bounds were proved for their length. It was noticed that, for these curves, the following property holds:

$$\text{if } t_1 < t_2 < t_3 \quad \text{then} \quad |x(t_1) - x(t_2)| \leq |x(t_1) - x(t_3)| \quad (3)$$

(with the opposite orientation a continuous curve satisfying (3) was called self contracting curve, see [6]. The authors [7] and [6] studied them and proved, for $n = 2$ bounds for their length). If a parametrization $x(\cdot)$ for the curve γ is available, the following extension of both above conditions (2),(3) will be used in this work:

$$\forall \Omega_\tau, \forall y \in \Omega_\tau, \text{ if } x(\tau) \notin \text{Int}(\Omega_\tau), \text{ then } \forall t > \tau : \quad |x(\tau) - y| \leq |x(t) - y|. \quad (4)$$

Sometimes, in the paper, in place of the quasi convex family $\{\Omega_\tau\}$, will be also considered a convex stratification \mathfrak{F} (definition 5.1). The class of convex stratifications (see [11]) is more general than the class of quasi convex families. A convex stratification is not necessarily parameterized by a continuous parameter. Special curves γ associated to a convex stratification \mathfrak{F} satisfying definition 6.3 (a generalization of (4)) are considered and studied. The couple (γ, \mathfrak{F}) is called **Expanding Couple** (EC). In the present work, two problems are addressed and solved.

First, properties of connected paths γ included in a convex body $\Omega \subset \mathbb{R}^n$ and satisfying (3) only (that will be called **Self Expanding Paths** (SEP)) are studied: the SEP turn out to be rectifiable curves with a priori bounded length, depending only on the dimension n and on the mean width of Ω (theorem 4.10).

Second, properties of convex stratifications \mathfrak{F} and expanding couples are considered. For any expanding couple (γ, \mathfrak{F}) regularity properties of the curve γ , associated to \mathfrak{F} , are studied. Existence, uniqueness and stability results, (theorems 6.15, 6.21) are proved for (γ, \mathfrak{F}) . Moreover γ can be parameterized in a such way that its representation $x(\cdot)$ is a lipschitz continuous time-dependent

trajectory of a differential inclusion of the type (2) (theorem 6.18). Also it is obtained that, if (γ, \mathfrak{F}) is an EC and $\Omega_1 \subset \Omega_2 \in \mathfrak{F}$ then, the length of the part of γ between Ω_1 and Ω_2 satisfies the bound:

$$\text{length}(\gamma \cap (\Omega_2 \setminus \Omega_1)) \leq \text{const} \cdot \text{dist}(\Omega_1, \Omega_2)$$

with Hausdorff distance and constant depending only on the dimension n (see theorem 6.7).

Results on the maximal length of a steepest descent curve for convex functions were noticed as an important tool for studying them (see [2]); the previous inequality provides an apriori bound for their lengths.

The main tools in our approach are: first, the suitable parametrization of the self expanding paths with respect to the mean width of the convex hulls of the increasing parts of the curve; second, the parametrization of the quasi convex families with respect to their mean width. The structure of the present work follows. In §2 and §3 preliminary facts are stated and properties on cap bodies and the variation of their mean width are studied. In §4 SEP are introduced and regularity results for them are proved. The main result in theorem 4.10 is that a SEP, with the mean width parametrization, is Lipschitz continuous. It is shown that its length can be bounded a priori. In §5 quasi convex stratifications and quasi convex families with their properties are considered. In §6 steepest descent curves and expanding couples are defined and studied; existence and uniqueness problems are stated and solved. Our approach about existence and regularity properties of steepest descent curves do not require that the time-dependent trajectory $x(\cdot)$ in (2) is absolutely continuous.

2 Preliminaries and definitions

Let

$$B(z, \rho) = \{x \in \mathbb{R}^n : |x - z| < \rho\}, \quad S^{n-1} = \partial B(0, 1) \quad n \geq 2.$$

A nonempty, compact convex set K of \mathbb{R}^n will be called a *convex body*. $\text{Int}(K)$ and ∂K denote the interior of K and the boundary of K , $\text{cl}(K)$ is the closure of K , $\text{Aff}(K)$ will be the smallest affine hull containing K , and $\text{relint } K$, $\partial_{\text{rel}} K$ are the corresponding subsets in the topology of $\text{Aff}(K)$. For any set S , $\text{co}(S)$ is the convex hull of S . $\text{Lin}^+(S)$ is the smallest linear space containing $S \cup \{0\}$, $\text{Lin}^-(S)$ is the largest linear space contained in $S \cup \{0\}$. $\text{Lin}^+, \text{Lin}^-$ operate in the vector space structure of \mathbb{R}^n . For a convex body $K \subset \mathbb{R}^n$, the *support function* is defined by

$$H_K(x) = \sup_{y \in K} \langle x, y \rangle, \quad x \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^n . The restriction of H_K to S^{n-1} will be denoted by h_K .

It is well known that, if $0 \leq \lambda \leq 1$, A and B are convex bodies, so is $\lambda A + (1 - \lambda)B$ and

$$H_{\lambda A + (1 - \lambda)B} = \lambda H_A + (1 - \lambda)H_B; \tag{5}$$

let us recall that H is monotone with respect to inclusion, i.e.

$$A \subseteq B \quad \text{if and only if} \quad H_A(x) \leq H_B(x) \quad \forall x \in \mathbb{R}^n. \tag{6}$$

The width of a convex set K in a direction $\theta \in S^{n-1}$ is the distance between the two hyperplanes orthogonal to θ and supporting K , given by $h_K(\theta) + h_K(-\theta)$. The *mean width* $w(K)$ of K is the mean of this distance on S^{n-1} with respect to the spherical Lebesgue measure σ , i.e.

$$w(K) = \frac{1}{\omega_n} \int_{S^{n-1}} (h_K(\theta) + h_K(-\theta)) d\sigma = \frac{2}{\omega_n} \int_{S^{n-1}} h_K(\theta) d\sigma \quad (7)$$

where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the measure of S^{n-1} .

Remark 2.1. If K is a convex body in \mathbb{R}^n and $k = \dim \text{Aff}(K) < n$, there are competing mean widths for K : $w_k(K)$ the mean width of K as subset of $\text{Aff}(K)$, $w_n(K)$ the mean width of K as subset of \mathbb{R}^n . Let us recall that $w_k(K)/w_n(K) = \frac{\omega_{k+1}}{\omega_k} \frac{\omega_n}{\omega_{n+1}}$ is a constant depending on n and k only.

In what follows $w(K)$ will always be $w_n(K)$.

Let K be a convex body and $q \in K$; the *normal cone* at q to K is the closed convex cone

$$N_K(q) = \{x \in \mathbb{R}^n : \langle x, y - q \rangle \leq 0 \quad \forall y \in K\}. \quad (8)$$

When $q \in \text{Int}(K)$ then $N_K(q)$ reduces to zero.

Definition 2.2. Let K be a convex body and p be a point not in K . A *simple cap body* K^p is:

$$K^p = \bigcup_{0 \leq \lambda \leq 1} \{\lambda K + (1 - \lambda)p\} = \text{co}(K \cup \{p\}). \quad (9)$$

Cap bodies properties can be found in [3]. For later use let us define also $K^p = K$ for $p \in K$.

Proposition 2.3. Let K be a convex body, $N_p = N_{K^p}(p)$ the normal cone to K^p at p , then

$$H_{K^p}(x) = \begin{cases} H_K(x) & \text{for } x \notin N_p, \\ \langle x, p \rangle & \text{for } x \in N_p. \end{cases} \quad (10)$$

Proof. Let $x \in N_p$, by (8)

$$\langle x, z \rangle \leq \langle x, p \rangle \quad \forall z \in K^p,$$

so $H_{K^p}(x) \leq \langle x, p \rangle$. Since $K^p \supset P = \{p\}$, monotone property (6) implies

$$H_{K^p}(x) \geq H_P(x) = \langle x, p \rangle \quad \forall x \in \mathbb{R}^n, \quad (11)$$

then, last part of (10) holds. If $x \notin N_p$, by (8) there exist $\bar{z} \in K$ and $\bar{\lambda} \in (0, 1]$, satisfying

$$\langle x, [\bar{\lambda}\bar{z} + (1 - \bar{\lambda})p] - p \rangle > 0,$$

i.e. $\langle x, \bar{z} - p \rangle > 0$. Then

$$H_K(x) > \langle x, p \rangle.$$

Let $z_1 \in K, \lambda_1 \in [0, 1]$ satisfying: $H_{K^p}(x) = \langle x, \lambda_1 z_1 + (1 - \lambda_1)p \rangle$. Then

$$H_{K^p}(x) \leq \max\{\langle x, z_1 \rangle, \langle p, x \rangle\} \leq H_K(x).$$

As $K^p \supset K$, $H_{K^p}(x) \geq H_K(x)$ and the first equality in (10) holds. \square

Let us notice that in previous proposition we don't assume that $p \notin K$, but later it will be used in that case.

The *dual cone* C^* of a convex cone C is

$$C^* = \{y \in \mathbb{R}^n : \langle y, x \rangle \geq 0 \quad \forall x \in C\}.$$

The dual cone C^* is a closed and convex cone.

In [12, theorem 5], it is observed that for a convex cone C

$$\dim \text{Lin}^+(C) + \dim \text{Lin}^-(C) = n.$$

The opening of a circular cone will be the amplitude of the acute angle between the axis and a generator half line. If C is a circular cone of opening α then C^* is a circular cone of opening $\pi/2 - \alpha$. The *tangent cone*, or support cone, of a convex body K at a point $q \in \partial K$ is given by

$$T_K(q) = \text{cl}\left\{\bigcup_{y \in K} \{s(y - q) : s \geq 0\}\right\}.$$

It is well known that:

$$N_K(q) = -T_K^*(q). \quad (12)$$

Moreover:

- (i) $\dim \text{Aff}(T_K(q)) = \dim \text{Aff}(K)$;
- (ii) $\dim \text{Lin}^-(N_K(q)) = n - \dim \text{Aff}(K)$;
- (iii) if $p \in \text{Aff}(K)$ then $\dim \text{Aff}(K^p) = \dim \text{Aff}(K)$;
if $p \notin \text{Aff}(K)$ then $\dim \text{Aff}(K^p) = \dim \text{Aff}(K) + 1$.

2.1 Sequences of cap bodies and their normal cones

Proposition 2.4. For $\varepsilon > 0$ let $p_\varepsilon \notin K$ and let $\lim_{\varepsilon \rightarrow 0^+} p_\varepsilon = p_0 \in \partial K$. Then

$$\limsup_{\varepsilon \rightarrow 0^+} N_{K^{p_\varepsilon}}(p_\varepsilon) \subseteq N_K(p_0). \quad (13)$$

Proof. The proposition is a consequence of standard results in the theory of convex analysis, however the proof is elementary, arguing by converging sequences. Let $\varepsilon_r \rightarrow 0^+$ for $r \rightarrow \infty$ and let for simplicity $p_r = p_{\varepsilon_r}$, $N_r = N_{K^{p_r}}(p_r)$, for $r \in \mathbb{N}$. Let $\{x_r\}_{r \in \mathbb{N}}$ be a converging sequence of points in N_r and $x_0 = \lim_{r \rightarrow \infty} x_r$. By construction

$$\langle x_r, y \rangle \leq \langle x_r, p_r \rangle \quad \forall y \in K^{p_r}, \forall r \in \mathbb{N}. \quad (14)$$

Since $K \subset K^{p_r}$, the previous inequality holds for all $y \in K$; going to the limit, one gets

$$\langle x_0, y \rangle \leq \langle x_0, p_0 \rangle \quad \forall y \in K.$$

This proves that $x_0 \in N_K(p_0)$. \square

Let $u \neq 0, u \in \mathbb{R}^n$. Let us consider the half space $\{u\}^* = \{x \in \mathbb{R}^n : \langle x, u \rangle \geq 0\}$, i.e. the dual cone of the half line starting from the origin through u .

Proposition 2.5. *Let $\varepsilon > 0$, K be a convex body and $p_0 \in \partial K$. Let $u \notin T_K(p_0)$ and let $p_\varepsilon = p_0 + \varepsilon u$. Then*

$$\{u\}^* \supset N_{K^{p_\varepsilon}}(p_\varepsilon) \supseteq N_K(p_0) \cap \{u\}^*.$$

Proof. As $u \notin T_K(p_0)$, then $p_\varepsilon \notin K$ and K^{p_ε} is a simple cap body. Let $x \in N_{K^{p_\varepsilon}}(p_\varepsilon)$, this means

$$\langle x, \lambda q + (1 - \lambda)p_\varepsilon - p_\varepsilon \rangle \leq 0, \quad \forall q \in K, \lambda \in [0, 1],$$

i.e.

$$\langle x, \lambda(q - p_0 - \varepsilon u) \rangle \leq 0. \quad (15)$$

Then $\langle x, u \rangle \geq 0$ so $x \in \{u\}^*$. If $x \in N_K(p_0)$, then $\langle x, q - p_0 \rangle \leq 0 \quad \forall q \in K$. If also $x \in \{u\}^*$, thus for every $\lambda \geq 0$, (15) is satisfied and $x \in N_{K^{p_\varepsilon}}(p_\varepsilon)$. \square

Theorem 2.6. *Let K be a convex body and $p_0 \in \partial K$. Let $u \notin T_K(p_0), u \neq 0$ and let $p_\varepsilon = p_0 + \varepsilon u$. Then $p_\varepsilon \notin K$ and*

$$\lim_{\varepsilon \rightarrow 0^+} N_{K^{p_\varepsilon}}(p_\varepsilon) = N_K(p_0) \cap \{u\}^*. \quad (16)$$

Proof. Propositions 2.4, 2.5 imply

$$\limsup_{\varepsilon \rightarrow 0^+} N_{K^{p_\varepsilon}}(p_\varepsilon) \subseteq N_K(p_0) \cap \{u\}^*. \quad (17)$$

Proposition 2.5 implies also $\liminf_{\varepsilon \rightarrow 0^+} N_{K^{p_\varepsilon}}(p_\varepsilon) \supseteq N_K(p_0) \cap \{u\}^*$. \square

3 First variation of the mean width of cap bodies

Let us recall that the Hausdorff distance between two convex bodies A and B can be written as

$$\text{dist}(A, B) = \max_{\theta \in S^{n-1}} |h_A(\theta) - h_B(\theta)|$$

(see [18, theorem 1.8.11]).

Let K be a convex body, $p_0 \in \partial K$. Let us consider the variation of K by deforming K as a simple cap body in a given direction $u \notin T_K(p_0)$. More precisely let $p_\varepsilon = p_0 + \varepsilon u$ as in proposition 2.5 and $N(p_0) = N_{K^{p_0}}(p_0)$, $N(p_\varepsilon) = N_{K^{p_\varepsilon}}(p_\varepsilon)$. When N is a cone let \hat{N} be the sector $N \cap S^{n-1}$.

Theorem 3.1. *Let $\varepsilon > 0$, $p_0 \in \partial K$, $u \notin T_K(p_0), u \neq 0$, $p_\varepsilon = p_0 + \varepsilon u$, then*

$$\frac{\omega_n}{2} (w(K^{p_\varepsilon}) - w(K)) = \varepsilon \int_{\widehat{N(p_0)} \cap \{u\}^*} \langle \theta, u \rangle d\sigma(\theta) + \int_{\widehat{N(p_\varepsilon)} \setminus (\widehat{N(p_0)} \cap \{u\}^*)} (h_{K^{p_\varepsilon}}(\theta) - h_K(\theta)) d\sigma(\theta). \quad (18)$$

The last integral of the right hand side is positive and infinitesimum of order greater than one for $\varepsilon \rightarrow 0^+$.

Proof. Let use formula (7) for the mean width. The integral of $(h_{K^{p_\varepsilon}}(\theta) - h_K(\theta))$ on S^{n-1} can be split on three sets

$$\widehat{N(p_0)} \cap \{u\}^*, \quad \widehat{N(p_\varepsilon)} \setminus (\widehat{N(p_0)} \cap \{u\}^*), \quad (S^{n-1} \setminus \widehat{N(p_\varepsilon)}) \cap \{u\}^*.$$

Since $K \subset K^{p_\varepsilon}$, by proposition 2.3 we have

$$h_{K^{p_\varepsilon}}(\theta) - h_K(\theta) = 0 \quad \text{for} \quad \theta \notin \widehat{N(p_\varepsilon)}.$$

Moreover for $\theta \in \widehat{N(p_0)} \cap \{u\}^*$ (which is included in $\widehat{N(p_\varepsilon)}$ by proposition 2.5), we have

$$h_{K^{p_\varepsilon}}(\theta) = h_{\{p_\varepsilon\}}(\theta) = \langle \theta, p_\varepsilon \rangle,$$

$$h_K(\theta) = h_{\{p_0\}}(\theta) = \langle \theta, p_0 \rangle.$$

Therefore

$$h_{K^{p_\varepsilon}}(\theta) - h_K(\theta) = \varepsilon \langle \theta, u \rangle \quad \text{for} \quad \theta \in \widehat{N(p_0)} \cap \{u\}^*,$$

and formula (18) is proved.

Since the Hausdorff distance between K^{p_ε} and K is less than $\varepsilon|u|$, then for any θ :

$$|h_{K^{p_\varepsilon}}(\theta) - h_K(\theta)| \leq \varepsilon|u|.$$

Theorem 2.6 implies that

$$\text{mis} \left(\widehat{N(p_\varepsilon)} \setminus (\widehat{N(p_0)} \cap \{u\}^*) \right) \rightarrow 0 \quad \text{for} \quad \varepsilon \rightarrow 0^+.$$

This proves that the last integral in (18) is infinitesimium of order greater than one of ε for $\varepsilon \rightarrow 0^+$ and it is positive since the cap body K^{p_ε} contains K . \square

The differential properties of $w(K^p)$ has been investigated in \mathbb{R}^3 in [19, Satz VI].

Proposition 3.2. *$w(K^p)$ is a convex function of p and for $p \notin \partial K$ is differentiable with*

$$\nabla w(K^p) = \frac{2}{\omega_n} \int_{\widehat{N(p)}} \theta \, d\sigma.$$

Proof. Let $p, q \in \mathbb{R}^n$; Let $p_\lambda = \lambda p + (1 - \lambda)q$, $0 \leq \lambda \leq 1$, then $K^{p_\lambda} \subseteq \lambda K^p + (1 - \lambda)K^q$, and $h_{K^{p_\lambda}} \leq \lambda h_{K^p} + (1 - \lambda)h_{K^q}$, therefore

$$w(K^{p_\lambda}) \leq \lambda w(K^p) + (1 - \lambda)w(K^q).$$

Hence $w(K^p)$ as a function of p is convex. If $p \in \text{Int}(K)$ then $N(p) = \{0\}$ and the thesis follows trivially. When $p \notin K$, from equality (18) (with K^p in place of K , p in place of p_0 , with n choices of vectors u linearly independents and not in $T_{K^p}(p)$) the thesis follows. \square

Next proposition is a consequence of corollary 2 in [20]. Let us give a shorter proof in our simpler situation of nested convex bodies.

Proposition 3.3. *For any two convex bodies $\Omega_1 \subset \Omega_2$ of \mathbb{R}^n , the following inequality holds*

$$\sqrt[n]{\frac{c_n^{(0)}}{(\text{diam}(\Omega_2))^{n-1}}} \text{dist}(\Omega_2, \Omega_1) \leq (w(\Omega_2) - w(\Omega_1))^{1/n}, \quad (19)$$

where $c_n^{(0)}$ depends only on n .

Proof. Let $\text{dist}(\Omega_2, \Omega_1) > 0$. Let $p \in \partial\Omega_2$, $q \in \partial\Omega_1$ such that $|p - q| = \text{dist}(\Omega_2, \Omega_1)$, and let p_0 be the mid point on the segment pq ; since

$$\Omega_2 \supset \Omega_1^p \supset \Omega_1^{p_0} \supset \Omega_1$$

then

$$w(\Omega_1^p) - w(\Omega_1^{p_0}) \leq w(\Omega_2) - w(\Omega_1). \quad (20)$$

Let u be the unit vector $(p - p_0)/|p - p_0|$ and let N_{p_0} be the normal cone at p_0 to $\Omega_1^{p_0}$. First let us observe that $N_{p_0} \subseteq \{u\}^*$. From (18)

$$w(\Omega_1^p) - w(\Omega_1^{p_0}) \geq \frac{2}{\omega_n} |p - p_0| \int_{\widehat{N(p_0)}} \langle \theta, u \rangle d\sigma(\theta).$$

The tangent cone at p_0 to $\Omega_1^{p_0}$ is contained in the circular cone $C_{u,\beta}$ with axis in direction of u and opening $\beta = \arctan(\frac{\text{diam}(\Omega_1)}{|q - p_0|})$. Therefore $N(p_0) \supseteq -C_{u,\beta}^* = C_{u,\pi/2-\beta}$, and from lemma 7.1

$$\int_{\widehat{N(p_0)}} \langle \theta, u \rangle d\sigma(\theta) \geq \int_{\widehat{C_{u,\pi/2-\beta}}} \langle \theta, u \rangle d\sigma(\theta) = \frac{\omega_{n-1}}{n-1} (\cos \beta)^{n-1}.$$

Since $2|q - p_0| = 2|p - p_0| = \text{dist}(\Omega_2, \Omega_1)$ and $\tan \beta = \frac{\text{diam}(\Omega_1)}{|p - p_0|}$, it follows that

$$\cos^2 \beta = (1 + \tan^2 \beta)^{-1} = \frac{|p - p_0|^2}{|p - p_0|^2 + \text{diam}^2(\Omega_1)} \geq \frac{|p - p_0|^2}{2(\text{diam}^2(\Omega_2))};$$

(19) follows from the previous three inequalities with $c_n^{(0)} = 2^{-(n-1)} \frac{\omega_{n-1}}{(n-1)\omega_n}$. \square

On the other hand from (7)

$$w(\Omega_2) - w(\Omega_1) \leq \frac{2}{\omega_n} \max_{S^{n-1}} |h_2(\theta) - h_1(\theta)| = \frac{2}{\omega_n} \text{dist}(\Omega_2, \Omega_1). \quad (21)$$

4 Self expanding paths

Definition 4.1. *Let us call self expanding path (SEP) a non empty, closed, connected and linearly strictly ordered (by \prec , with $x_1 \prec x_2 \Rightarrow x_1 \neq x_2$) subset γ of \mathbb{R}^n , with the property:*

$$x_1, x_2, x_3 \in \gamma, \quad \text{and} \quad x_1 \prec x_2 \prec x_3 \quad \implies \quad |x_2 - x_1| \leq |x_3 - x_1|. \quad (22)$$

This class of paths, was studied as class of curves in [15] with an added rectifiability hypothesis; differential properties and bounds were obtained. Here properties are proved from the above geometric definition with no rectifiability assumptions.

Remark 4.2. *Let us notice that the graph of a continuous and monotone real function f on a bounded interval is a SEP with ordering $p_1 \equiv (t_1, f(t_1)) \prec p_2 \equiv (t_2, f(t_2))$ iff $t_1 < t_2$.*

Definition 4.3. *Let $x_0 \in \gamma$, let us denote*

$$\gamma_{x_0} = \{x \in \gamma : x \prec x_0\} \cup \{x_0\}.$$

Proposition 4.4. *If γ is a self expanding path and $x \in \gamma$, then for any $p, q \in co(\gamma_x) \setminus \{x\}$*

$$\langle p - x, q - x \rangle > 0, \quad (23)$$

and any two half lines from $x \in \gamma$ in the tangent cone at $co(\gamma_x)$ are the sides of an angle less than or equal to $\pi/2$.

Proof. It is enough to prove inequality (23) for $x_1, x_2 \in \gamma$, $x_1 \prec x_2 \prec x$. From (22)

$$0 < |x_2 - x_1| \leq |x - x_1|;$$

therefore the triangle of vertices x, x_1, x_2 has an acute angle at the vertex x . \square

Corollary 4.5. *At any point p of any self expanding path γ the inclusion*

$$N_{co(\gamma_p)}(p) \supseteq -T_{co(\gamma_p)}(p) \quad (24)$$

holds.

From now on it will be assumed that all the self expanding paths γ considered are contained in a closed ball \tilde{B} , and let Γ this class. Of course if $\gamma \in \Gamma$ and $x_0 \in \gamma$, then $\gamma_{x_0} \in \Gamma$. The path γ with the topology induced by \mathbb{R}^n is a metric space. Let us notice also that $Int(co(\gamma_x))$ can be an empty set, i.e $\dim Aff(co(\gamma_x)) < n$.

Lemma 4.6. *Let $\gamma \in \Gamma$. The following properties hold*

(i) *if $x \in \gamma$, $x \neq \min \gamma$, then $x \in \partial_{rel} co(\gamma_x)$;*

(ii) *$x_1 \prec x_2 \in \gamma \Rightarrow x_2 \notin co(\gamma_{x_1})$;*

(iii) *$x_1 \prec x_2 \in \gamma \Rightarrow w(co(\gamma_{x_1})) < w(co(\gamma_{x_2}))$.*

Proof. (i) follows from proposition 4.4: if x is not on the relative boundary of $co(\gamma_x)$ then the tangent cone at x will be $Aff(co(\gamma_x))$, in contradiction with (23). If (ii) does not hold, then $x_2 \in co(\gamma_{x_1})$. On the other hand x_2 has positive distance from the compact set γ_{x_1} ; then x_2 must be in the interior of a segment with end points $y, z \in co(\gamma_{x_1}) \subset co(\gamma_{x_2})$, in contradiction with (23). (iii) follows from (ii) since $co(\gamma_{x_1}) \subsetneq co(\gamma_{x_2})$ and the mean width is a strictly increasing function in the class of convex bodies with respect the inclusion relation. \square

Definition 4.7. Let us call a parametrization of a path $\gamma \in \Gamma$ a mapping of a real interval T , $T \ni t \rightarrow x(t) \in \gamma$, satisfying $\forall x_0 \in \gamma$, $\{t \in T : x(t) = x_0\}$ is an interval (possibly reduced to a point), and

$$x_0, x_1 \in \gamma, x_0 \prec x_1 \implies \sup\{t \in T : x(t) = x_0\} < \inf\{t \in T : x(t) = x_1\}.$$

A parametrization will be called continuous if T is a closed interval and $x(\cdot)$ is continuous in T .

Theorem 4.8. Let $\gamma \in \Gamma$. The path γ has a one-to-one continuous parametrization $x(w)$, inverse of

$$w(x) := w(\text{co}(\gamma_x)) \in [0, w(\text{co}(\gamma))]. \quad (25)$$

Proof. From (iii) of previous lemma the map $w : \gamma \ni x \rightarrow w \in [0, w(\text{co}(\gamma))]$ is injective. Moreover if $x_1 \prec x_2$, $|x_2 - x_1| < \varepsilon$ then the ball $B(x_1, \varepsilon)$ contains all the points of γ between x_1 and x_2 . Therefore

$$\text{co}(\gamma_{x_2}) \subset \text{co}(\gamma_{x_1}) + B(x_1, \varepsilon).$$

Hence $w(\text{co}(\gamma_{x_2})) - w(\text{co}(\gamma_{x_1})) \leq 2\varepsilon$. Then, if γ is equipped with the topology induced by \mathbb{R}^n , the map w is continuous and maps γ in a connected subset of $[0, w(\text{co}(\gamma))]$. Since γ is compact w has minimum and maximum; by (iii) they are 0, and $w(\text{co}(\gamma))$ respectively. Thus w is bijective and continuous; its inverse:

$$[0, w(\text{co}(\gamma))] \ni w \rightarrow x(w)$$

is a continuous parametrization of γ . \square

Let γ be a self expanding path with a continuous parametrization $x(\cdot)$ defined in a real interval T . Let $\gamma(t) = \gamma_{x(t)}$. Let us notice that the integer valued function $T \ni t \rightarrow \dim \text{Aff}(\text{co}(\gamma(t)))$ is not decreasing and left continuous.

The following property comes out from elementary geometry and will be used later.

Lemma 4.9. Let $p, q, y_i \in \mathbb{R}^n$, $i = 1, \dots, s$. If

$$|p - y|^2 \leq |q - y|^2, \quad \text{for } y = y_i, i = 1, \dots, s \quad (26)$$

then the same holds for any $y \in \text{co}(\{y_i, i = 1, \dots, s\})$.

Proof. It is enough to prove that if (26) holds for y_1, y_2 then, it holds with any y on the line segment $y_1 y_2$. Now (26) is equivalent to claim that both y_i , $i = 1, 2$ belong to the half space containing p and delimited by the hyperplane orthogonal to the line segment pq at the middle point of it. Since such half space is convex, it contains all the points y on $y_1 y_2$. Therefore the inequality (26) holds for all $y \in y_1 y_2$. \square

$||\gamma|| = \int_T |\dot{x}(t)| dt$ denotes the length of a rectifiable curve γ , with a parametrization $T \ni t \rightarrow x(t)$.

Theorem 4.10. Let $\gamma \in \Gamma$ be a self expanding path in \mathbb{R}^n . Then γ , parameterized by the mean width function (25) is Lipschitz continuous, and a.e.

$$\left| \frac{dx}{dw} \right| \leq c_n^{(1)}, \quad (27)$$

where $c_n^{(1)}$ is a constant depending only on the dimension n . In particular any self expanding path γ (which is connected) is a rectifiable curve and

$$||\gamma|| \leq c_n^{(1)} \cdot w(\text{co}(\gamma)). \quad (28)$$

Proof. Step a): Let $x(w), w \in [0, w(\text{co}(\gamma))]$ be the continuous parametrization of γ introduced in theorem 4.8. Let $0 = w_0 < \dots < w_i < \dots < w_s = w(\text{co}(\gamma))$ the decomposition of $[0, w(\text{co}(\gamma))]$ such that if $w \in (w_i, w_{i+1}]$, $i = 0, \dots, s-1$ then, $\dim \text{Aff}(\text{co}(\gamma(w)))$ has constant value m_i . It is sufficient to prove that $x(w)$ is lipschitz continuous in $[w_i, w_{i+1}]$ and (27) holds. If $m_0 = 1$, $[w_0, w_1] \ni w \rightarrow x(w)$ is linear and its derivative with respect to the one dimensional mean width is trivially 1; then, (27) holds in $[w_0, w_1]$ with $c_n^{(1)} = \pi \frac{\omega_n}{\omega_{n+1}}$.

Step b): It can be assumed that $m_i \geq 2, i = 0, \dots, s-1$. For every $\bar{w} \in (w_i, w_{i+1})$, the relative interior of $\text{co}(\gamma(\bar{w}))$ is non empty; let $B(\tilde{y}, \rho_1)$ a m_i -dimensional ball such that

$$B(\tilde{y}, \rho_1) \subset \text{relint } \text{co}(\gamma(\bar{w})).$$

Let us fix $\bar{w} \in (w_i, w_{i+1})$. Then for every $w' \in [\bar{w}, w_{i+1}]$

$$B(\tilde{y}, \rho_1) \subset \text{relint } \text{co}(\gamma(\bar{w})) \subset \text{relint } \text{co}(\gamma(w')).$$

In the remaining part of this step and in the steps c) and d), for simplicity, we will be arguing with $m_i = n$.

Let $p' = x(w')$, $K_{p',1}$ be the convex cone with opening $\alpha = \alpha(w')$ such that $p' + K_{p',1}$ is tangent to the ball $B(\tilde{y}, \rho_1)$; from proposition 4.4 it follows that $0 < \alpha \leq \pi/4$ and

$$0 < \rho_1 \leq \frac{\tilde{y} - p'}{\sqrt{2}} \leq \frac{\text{diam}(\text{co}(\gamma))}{\sqrt{2}}.$$

Let $K_{p',1/2}$ be the convex cone, with the same axis as $K_{p',1}$ and opening $\alpha/2$. Then

$$K_{p',\lambda} \subset T_{\text{co}(\gamma(w'))}(p'), \quad \text{for } \lambda = 1, \frac{1}{2}, \quad w' \in [\bar{w}, w_{i+1}].$$

As a consequence of (24)

$$-K_{p',1} \subset N_{\text{co}(\gamma(w'))}(p') = -(T_{\text{co}(\gamma(w'))}(p'))^* \subset -(K_{p',1/2})^*. \quad (29)$$

Moreover if a unit vector $u \in -(K_{p',1/2})^*$ then

$$\{u\}^* \supset -K_{p',1/2}, \quad (30)$$

i.e.

$$\langle u, \theta \rangle \geq 0 \quad \text{if } \theta \in -K_{p',1/2}, \quad (31)$$

and as consequence $u \notin T_{\text{co}(\gamma(w'))}(p')$.

Step c): The aim of this step is to prove that there exists a constant $C = C(\rho_1)$ so that, for every $w'' \in [w', w_{i+1}]$,

$$|x(w'') - x(w')| < C \implies x(w'') - x(w') \in -(K_{p',1/2})^*. \quad (32)$$

Let $y \in \gamma(\bar{w})$. As γ is a SEP, the real function $w \rightarrow |x(w) - y|$ is not decreasing for $w \geq \bar{w}$; then from lemma 4.9 the same holds for every $y \in \text{co}(\gamma(\bar{w}))$; in particular

$$|x(w'') - y| \geq |x(w') - y| = |p' - y|, \quad \forall y \in B(\tilde{y}, \rho_1), \quad w'' \geq w' \geq \bar{w}. \quad (33)$$

Let $p'' = x(w'')$. The inequalities (33) imply:

$$p'' \notin \Phi_{p'} := \bigcup_{v \in \partial(p' + K_{p',1}) \cap \partial B(\tilde{y}, \rho_1)} B(v, |p' - v|).$$

The boundary of $p' - (K_{p',1/2})^*$ intersects the boundary of $\Phi_{p'}$ in a $(n-2)$ -dimensional sphere whose points have distance from p' given by $l = \rho_1 \frac{\cos \alpha}{\cos \alpha/2}$, (see Figure 1). If p'' has distance from p' less

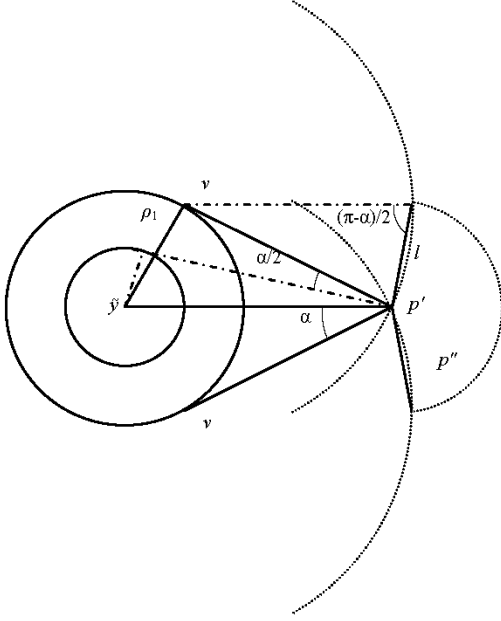


Figure 1: The boundary of $\Phi_{p'}$.

than l , then $p'' \in p' - (K_{p',1/2})^*$, i.e.

$$|p'' - p'| < \rho_1 \frac{\cos \alpha}{\cos \alpha/2} \implies p'' - p' \in -(K_{p',1/2})^*.$$

Since $\phi(\alpha) = \frac{\cos \alpha}{\cos \alpha/2}$ is decreasing in $[0, \pi/4]$, then (32) is proved with

$$C = \rho_1 \cdot \min \phi = \rho_1 \frac{\cos \pi/4}{\cos \pi/8}.$$

Step d): The goal of this step is to prove that $x(w)$ is locally lipschitz continuous in the open interval (w_i, w_{i+1}) with $n \geq 2$. Let $\delta > 0$; uniform continuity of $x(\cdot)$ in $[\bar{w}, w_{i+1}]$ guarantees that there exists h_0 such that $0 < h < h_0$ implies that $|x(w' + h) - x(w')| < \delta$. Let us choose $w'' = w' + h$, $\delta = C(\rho_1)$, then u , the unit direction of $x(w' + h) - x(w')$, belongs to $-(K_{p',1/2})^*$. As $|p'' - p'| < \delta$,

from the previous step, (31) holds and $u \notin T_{co(\gamma(w'))}(p')$; thus $\{u\}^* \cap N_{co(\gamma(w'))}(p') \supseteq -(K_{p',1/2})$. Let us notice that (using the cap body notation)

$$(co(\gamma(w')))^{p''} \subset co(\gamma(w'')).$$

Since (31) holds, theorem 3.1 can be applied with $K = co(\gamma(w'))$, $p_0 = p'$, $p_\varepsilon = p''$ and u as above. It follows that

$$(w' + h) - w' \geq w(K^{p''}) - w(K) \geq |p'' - p'| \frac{2}{\omega_n} \int_{\widehat{N(p') \cap \{u\}^*}} \langle \theta, u \rangle d\sigma(\theta), \quad (34)$$

with $N_{co(\gamma(w'))}(p') = N(p')$. From (29) we have $N_{p'} \supseteq -K_{p',1/2}$, and with (30) we get

$$\int_{\widehat{N(p') \cap \{u\}^*}} \langle \theta, u \rangle d\sigma \geq \int_{-\widehat{K_{p',1/2}}} \langle \theta, u \rangle d\sigma.$$

From lemma 7.2 in the appendix, last integral is bounded from below from a positive constant $C(n, \alpha) = \frac{\omega_{n-1}}{n-1} \cdot \sin^n(\frac{\alpha}{4})$; since $C(n, \alpha)$ is increasing for $0 < \alpha \leq \pi/4$ and α is greater than $\bar{\alpha} = \arcsin(\rho_1/\text{diam}(co(\gamma)))$, then $C(n, \alpha) > C(n, \bar{\alpha})$, and this bound is uniform in $[\bar{w}, w_{i+1}]$. This fact proves that $x(w)$ is lipschitz continuous in $[\bar{w}, w_{i+1}]$ with a constant depending on $\bar{w}, \text{diam}(\gamma)$.

Step e): As \bar{w} is arbitrary in (w_i, w_{i+1}) , $x(w)$ is locally lipschitz in that interval, thus rectifiable. By using [15, theorem VII] in every closed subinterval of (w_i, w_{i+1}) ,

$$|x(w'') - x(w')| \leq c(m_i)|w'' - w'|, \quad (35)$$

holds, where $c(m_i)$ depends on m_i only. As a consequence (35) holds in $[w_i, w_{i+1}]$ and (27) follows. \square

In [15] it has been proved that $c_2^{(1)} = \pi$ (best possible constant) and

$$c_n^{(1)} \leq (n-1) \cdot n^{n/2} \frac{\omega_n}{\omega_{n-1}}.$$

If we drop in the definition 4.1 the assumption that γ is connected, then the continuity of the inverse of the map w in theorem 4.8 does not hold. As example the piecewise steepest descent curves considered in [2, definition 15] are not connected, moreover it was proved that they are not rectifiable. Here only connected curves are considered; of course, if γ is not connected, the corresponding properties hold for each connected component of γ .

Let us notice that there exist SEP with not absolutely continuous parametrization. To provide an example let us consider the Cantor function $[0, 1] \ni t \rightarrow g(t)$, see [13, p. 83], which is a not decreasing function, with zero derivative a.e. in $[0, 1]$, not absolutely continuous, with $g(0) = 0, g(1) = 1$. The graphic curve $\zeta : [0, 1] \ni t \rightarrow (t, g(t))$ is a planar SEP, but it is not absolutely continuous, in particular with this parametrization ζ is not Lipschitz.

Theorem 4.11. *Let γ be a self expanding path and let $x(\cdot)$ be a continuous parametrization of γ defined in a real interval T . Then,*

$$|x(t'') - x(t')| \leq |x(t) - x(t')| \quad \text{for all} \quad t' \leq t'' \leq t \in T. \quad (36)$$

Moreover where $\dot{x}(t)$ exists, the property

$$\dot{x}(t) \in N_{co(\gamma(t))}(x(t)) \quad (37)$$

holds. If $x(\cdot)$ is parameterized by using the curvilinear abscissa s , the formula

$$\frac{dw(co(\gamma(s)))}{ds} \geq \frac{2}{\omega_n} \int_{N(x(s))} \langle \theta, x'(s) \rangle d\sigma \quad (38)$$

holds.

Proof. (36) follows from (22). In other words, for all $t' \in T$ the function

$$F(\cdot, x(t')) : t \rightarrow |x(t) - x(t')|^2 \quad (39)$$

is a not decreasing function in $t \geq t' \in T$, with derivative greater than zero a.e., this implies (37). Inequality (38) it is in ([15], theorem VII). It follows also from (18). \square

Remark 4.12. Using (18) it can be proved that actually equality holds a.e. in (38).

5 Quasi convex families

Nested families of convex sets have been studied by De Finetti [11] and Fenchel [12].

Definition 5.1. Let us call **convex stratification**, (see [11]), a non empty family \mathfrak{F} of convex bodies in \mathbb{R}^n , linearly strictly ordered by inclusion ($\Omega_1 \subset \Omega_2$, $\Omega_1 \neq \Omega_2$), with a maximum set and a minimum set, not identical.

Let us call a parametrization of \mathfrak{F} the inverse of a strictly increasing map of \mathfrak{F} into a subset $W_{\mathfrak{F}}$ of a compact interval $T \subset \mathbb{R}$.

Definition 5.2. Let \mathfrak{F} be a parameterized convex stratification with a parametrization satisfying $W_{\mathfrak{F}} \equiv T$. The family \mathfrak{F} will also be denoted $\{\Omega_t\}_{t \in T}$. If for every $s \in T \setminus \{\max T\}$ the property:

$$\bigcap_{t > s} \Omega_t = \Omega_s$$

holds, then as in [12], $\{\Omega_t\}_{t \in T}$ will be called a **quasi convex family**.

In [12] was noticed that $\{\Omega_t\}_{t \in T}$ is a quasi convex family iff there exists a lower semi continuous quasi convex function, with $\{\Omega_t\}_{t \in T}$ the family of its sub level sets.

Let $\mathfrak{F}, \mathfrak{G}$ be two convex stratifications. Let us say that \mathfrak{F} is contained in \mathfrak{G} if every element of \mathfrak{F} is an element of \mathfrak{G} .

Definition 5.3. A quasi convex family \mathfrak{F} will be called **connected** if

$$\forall x \in \max \mathfrak{F} \setminus \text{relint} \min \mathfrak{F} \quad \exists Q \in \mathfrak{F} : x \in \partial_{rel} Q.$$

Let \mathfrak{F} be a connected quasi convex family; in \mathfrak{F} let us consider the usual Hausdorff distance between compact sets. A parametrization of \mathfrak{F} will be called continuous if the map from the metric space \mathfrak{F} to $T = W_{\mathfrak{F}}$ is continuous.

Remark 5.4. *Let us notice that a quasi convex family may be not connected. As example let us consider the family $\{\Omega_\tau\}$ of the sub level sets of a continuous quasi convex function f with a flat zone; a flat zone for f is an annulus with interior points bounded by two convex bodies, where f is constant. For any interior point x in the annulus does not exist a level subset Ω_τ with $x \in \partial_{rel}\Omega_\tau$.*

In [12] it is noticed that

$$\Omega_s = \overline{\cup_{t < s} \Omega_t} \quad \forall s \in T \setminus \{\min T\}. \quad (40)$$

is a necessary condition for \mathfrak{F} to be a family of sublevel sets for a convex function.

Lemma 5.5. *A quasi convex family $\mathfrak{F} = \{\Omega_t\}_{t \in T}$ is connected iff (40) holds.*

Proof. (40) \Rightarrow \mathfrak{F} is connected.

Let $x \in \max \mathfrak{F} \setminus \text{relint } \min \mathfrak{F}$. If $\{t \in T : \text{relint } \Omega_t \ni x\}$ is empty, then $x \in \partial_{rel} \max \mathfrak{F}$; if not, let $t_2 := \inf\{t \in T : \text{relint } \Omega_t \ni x\}$; then $\Omega_{t_2} = \cap_{t > t_2} \Omega_t \ni x$. If $t_2 = \min T$, then $x \notin \text{relint } \min \mathfrak{F}$, and $x \in \partial_{rel} \min \mathfrak{F}$. If $t_2 > \min T$, then by (40),

$$\Omega_{t_2} = \overline{\cup_{t < t_2} \Omega_t};$$

as $\Omega_t \not\ni x$ ($t < t_2$), then $x \notin \cup_{t < t_2} \Omega_t$ and thus $x \in \partial_{rel} \Omega_{t_2}$.

\mathfrak{F} is connected \Rightarrow (40).

Assume, by contradiction, that there exists $s_0 \in T \setminus \{\min T\}$ satisfying

$$\overline{\cup_{t < s_0} \Omega_t} \subset \Omega_{s_0} \quad \text{and} \quad \overline{\cup_{t < s_0} \Omega_t} \neq \Omega_{s_0};$$

then $\text{relint } \Omega_{s_0} \setminus (\overline{\cup_{t < s_0} \Omega_t}) \neq \emptyset$. Thus there exists $x_0 \in \text{relint } \Omega_{s_0}$, $x_0 \notin \Omega_t$ ($t < s_0$), $x_0 \in \text{relint } \Omega_t$ ($t > s_0$), contradicting the hypothesis that \mathfrak{F} is connected. \square

Lemma 5.6. *Let $\mathfrak{F} = \{\Omega_t\}_{t \in T}$ be a connected quasi convex family and let K be a convex body satisfying*

$$\min \mathfrak{F} \subseteq K \subseteq \max \mathfrak{F}.$$

Then, there are $\Omega_{t_1}, \Omega_{t_2} \in \mathfrak{F}$ satisfying $\Omega_t \supseteq K$ iff $t \geq t_2$, and $\Omega_t \subseteq K$ iff $t \leq t_1$.

Proof. If $K = \min \mathfrak{F}$ or $K = \max \mathfrak{F}$ the lemma is obvious. Let us assume $\min \mathfrak{F} \neq K \neq \max \mathfrak{F}$. Let $t_2 = \inf\{t \in T : \Omega_t \supseteq K\}$; then $\Omega_{t_2} = \cap_{t > t_2} \Omega_t \supseteq K$. Moreover $\Omega_t \supseteq K$ if and only if $t \geq t_2$. Let $t_1 = \sup\{t \in T : \Omega_t \subseteq K\}$ and $A = \bigcup_{t \in T} \{\Omega_t \subseteq K\}$; then $\Omega_{t_1} \supseteq \text{cl } A$ and $\Omega_t \subseteq K$ is not possible if $t > t_1$. As \mathfrak{F} is connected, by previous lemma $\Omega_{t_1} = \text{cl } A$, thus $\Omega_t \subseteq K$ if and only if $t \leq t_1$. \square

Theorem 5.7. *Let \mathfrak{F} be a connected quasi convex family, then \mathfrak{F} , with the Hausdorff distance, is a connected complete metric space; moreover the mean width parametrization: $w = w(K)$, $K \in \mathfrak{F}$, is a continuous parametrization on the compact interval $[w(\min \mathfrak{F}), w(\max \mathfrak{F})]$. On the other hand if \mathfrak{F} is a convex stratification and $w(\mathfrak{F}) = [w(\min \mathfrak{F}), w(\max \mathfrak{F})]$, then \mathfrak{F} is a connected quasi convex family.*

Proof. The family \mathfrak{K} of all compact convex subsets of $\max \mathfrak{F}$, with the Hausdorff distance is a complete metric space by Blaschke's selection theorem (see e.g. [3]). Let $K \in \mathfrak{K}$, $\Omega^{(l)} \in \mathfrak{F}$ such that

$$\lim_{l \rightarrow \infty} \text{dist}(\Omega^{(l)}, K) = 0. \quad (41)$$

Let us show that $K \in \mathfrak{F}$. Let $\Omega_{t_1}, \Omega_{t_2}$ as in the previous lemma. If $\text{dist}(\Omega_{t_2}, K) = 0$ then $K = \Omega_{t_2} \in \mathfrak{F}$; similarly if $\text{dist}(\Omega_{t_1}, K) = 0$ then $K = \Omega_{t_1} \in \mathfrak{F}$. From (41), the case that $\text{dist}(\Omega_{t_2}, K) > 0$ and $\text{dist}(\Omega_{t_1}, K) > 0$ cannot occur.

The mean width parametrization $w(K)$, $K \in \mathfrak{F}$ is a strictly increasing from the connected strictly linearly ordered set \mathfrak{F} to $W_{\mathfrak{F}} = [w(\min \mathfrak{F}), w(\max \mathfrak{F})]$; since the Hausdorff distance on the elements of \mathfrak{F} and the mean width distance (see (21) and proposition 3.3) are equivalent, then $w : \mathfrak{F} \rightarrow W_{\mathfrak{F}}$ is a one to one, strictly increasing function and its inverse is a continuous parametrization of \mathfrak{F} . If \mathfrak{F} is a convex stratification and $w(\mathfrak{F}) = [w(\min \mathfrak{F}), w(\max \mathfrak{F})]$ then \mathfrak{F} , with the parameter w , is a quasi convex family. The final part of the theorem follows by lemma 5.5. \square

Theorem 5.8 (of completeness). *Let \mathfrak{F} be a convex stratification. Then, there exists a connected quasi convex family \mathfrak{G} containing \mathfrak{F} so that $\min \mathfrak{G} = \min \mathfrak{F}$, $\max \mathfrak{G} = \max \mathfrak{F}$.*

Proof. Let us parameterize the elements of the given family \mathfrak{F} by their mean width parameter $\tau = w(\Omega_{\tau})$, for $\Omega_{\tau} \in \mathfrak{F}$. Let $\Sigma := [w(\min \mathfrak{F}), w(\max \mathfrak{F})]$, then

$$\Omega_{\tau_1} \subset \Omega_{\tau_2}, \Omega_{\tau_1} \neq \Omega_{\tau_2} \quad \text{iff} \quad \tau_1, \tau_2 \in w(\mathfrak{F}), \tau_1 < \tau_2.$$

If $\Sigma \setminus w(\mathfrak{F}) = \emptyset$ the theorem is proved. If $w(\mathfrak{F})$ is not closed, let $s \in \text{cl}(w(\mathfrak{F})) \setminus w(\mathfrak{F})$. Let us add to \mathfrak{F} the convex body (that will be called Ω_s), obtained by limit of convex bodies of \mathfrak{F} . This is well defined, since \mathfrak{F} from the previous theorem is a subset of the complete metric space of all compact convex subsets of $\max \mathfrak{F}$. Let us close \mathfrak{F} according to this topology and let us call again \mathfrak{F} the new completed family. The function w can be extended in a continuous way to the augmented family \mathfrak{F} . If $\Sigma \setminus w(\mathfrak{F}) = \emptyset$ the theorem is proved. If not, $w : \mathfrak{F} \rightarrow \Sigma$ is a strictly increasing continuous function and $\Sigma \setminus w(\mathfrak{F})$ is union of numerable relatively open intervals with end points in $w(\mathfrak{F})$. Let $\tau \in \Sigma \setminus w(\mathfrak{F})$. Let (τ_1, τ_2) the maximal interval enclosed in $\Sigma \setminus w(\mathfrak{F})$ containing τ . Then let us define for $\tau_1 < \lambda < \tau_2$ the interpolation between the convex sets $\Omega_{\tau_1}, \Omega_{\tau_2}$:

$$A_{\lambda} = \{x \in \Omega_{\tau_2} : \text{dist}(x, \Omega_{\tau_1}) \leq \text{dist}(\Omega_{\tau_1}, \Omega_{\tau_2}) \frac{\lambda - \tau_1}{\tau_2 - \tau_1}\}. \quad (42)$$

The convex set A_{λ} is the intersection between Ω_{τ_2} and the parallel convex body to Ω_{τ_1} , at distance $(\lambda - \tau_1)/(\tau_2 - \tau_1)$ from Ω_{τ_1} . For $\tau_1 < \tau < \tau_2$ let

$$\Omega_{\tau} := A_{\lambda} \quad \text{iff} \quad w(A_{\lambda}) = \tau,$$

and let us add these sets to the initial family. Let $\mathfrak{G} := \{\Omega_{\tau}\}_{\tau \in \Sigma}$ be the augmented family. \mathfrak{G} is parameterized by its mean width parameter and $w(\mathfrak{G}) = [w(\min \mathfrak{G}), w(\max \mathfrak{G})]$; then, by previous theorem, \mathfrak{G} is a connected quasi convex family. \square

Definition 5.9. Let \mathfrak{K} be the space of all compact convex subsets of Ω equipped with the Hausdorff distance. Let $\mathfrak{G}^{(m)} = \{\Omega_w^{(m)}\}_{w \in [w(\Omega_0), w(\Omega)]}$ a sequence of connected quasi convex families parameterized by the mean width w , satisfying $\min \mathfrak{G}^{(m)} = \Omega_0$ and $\max \mathfrak{G}^{(m)} = \Omega$. Let us define

$$\lim_{m \rightarrow \infty} \mathfrak{G}^{(m)} = \mathfrak{G} = \{\Omega_w\}_{w \in [w(\Omega_0), w(\Omega)]}$$

if the continuous functions $w \rightarrow \Omega_w^{(m)}$ uniformly converge to $w \rightarrow \Omega_w$.

6 Steepest descent curves for quasi convex families

Let u be a smooth function defined in a convex body Ω . Let $Du(x) \neq 0, \forall x \in \Omega : u(x) > \min u$. A classical steepest descent curve of u is a rectifiable curve $s \rightarrow x(s)$ solution to

$$\frac{dx}{ds} = \frac{Du}{|Du|}(x(s))$$

(some authors call them steepest descent curves with ascent parameter or steepest ascent curves). Classical steepest descent curves are the integral curves of a unit field normal to the sub level sets of the given function u . Here we are interested to convex sub levels, i.e. when u is a quasi convex function. Let us consider the family of the sub level sets of u : $\Omega_t = \{x \in \Omega : u(x) \leq t\}$; let us notice that the family $\{\Omega_t\}$ is a connected quasi convex family.

Let us give now an extended definition of a steepest descent curve related to a connected quasi convex family.

Definition 6.1. Let T be a closed real interval and let $\{\Omega_t\}_{t \in T}$ be a connected quasi convex family. A continuous path $t \rightarrow x(t)$ will be called a **viable steepest descent curve** for $\{\Omega_t\}_{t \in T}$ if

$$(i) \quad x(t) \in \partial_{rel} \Omega_t \quad \forall t \in T \setminus \{\min T\};$$

(ii) $t \rightarrow x(t)$ is a solution in T of the differential inclusion problem:

$$\dot{x}(t) \in N_{\Omega_t}(x(t)) \quad \text{a.e. in } T. \quad (43)$$

$x(\max T)$ will be called the end point of the steepest descent curve.

Every self expanding path γ parameterized with arc length s is a viable steepest descent curve for the family $\{co(\gamma(s))\}, 0 \leq s \leq \|\gamma\|$, see (37).

For suitable quasi convex families is not possible to get existence results of viable steepest descent curves, as the following example in \mathbb{R}^3 shows.

Example 6.2. Let $E_t, t \in (1, 2]$ be the family of convex sets, defined by the rotations around the x_3 axis of the following plane sets: union of the semicircles

$$(x_1 - 1)^2 + x_3^2 \leq (t - 1)^2, \quad x_1 \geq 1$$

and rectangles

$$|x_3| \leq t - 1, \quad 0 \leq x_1 \leq 1.$$

Let $D_t, t \in [0, 1]$ be the family of circles in the plane $x_3 = 0$

$$x_1^2 + x_2^2 \leq t^2.$$

The family $\{\Omega_t\}_{t \in [0,2]} = \{D_t\}_{t \in [0,1]} \cup \{E_t\}_{t \in (1,2]}$ is a connected quasi convex family. Any viable steepest descent curve $x(\cdot)$ of $\{E_t\}_{t \in (1,2]}$ with end point $(\overline{x}_1, \overline{x}_2, \pm 1) \in \partial E_2$ is a segment; moreover if $\overline{x}_1^2 + \overline{x}_2^2 < 1$, then $x(\cdot)$ stops at time $t = 1$ at a point $x(1) \in \text{relint } D_1$. Any continuous extension of $x(\cdot)$ to the interval $[0, 1]$ as SEP of $\{\Omega_t\}_{t \in [0,2]}$ does not satisfy (i) in the definition 6.1 for $t \in (\tau, 1]$ with $\tau = |x(1)|^2$; then it is not a viable steepest descent curve.

Let us give a definition both extending the viable steepest descent curves of the definition 6.1 and generalizing the class of SEP of the previous section. The aim of the following definition is to bind the natural order structure of a quasi convex stratification with the natural order structure of an associated self expanding path.

Definition 6.3. Let \mathfrak{F} be a convex stratification and let γ be a self expanding path enclosed in $\max \mathfrak{F}$. The couple (γ, \mathfrak{F}) will be called an **expanding couple** (EC) if:

- (i) $\forall Q \in \mathfrak{F}, \gamma \cap Q \neq \emptyset$,
- (ii) $\gamma \cap \partial_{\text{rel}} \max \mathfrak{F} \neq \emptyset$,
- (iii) $\forall Q \in \mathfrak{F}, \forall y \in Q, \forall x \in \gamma : x \notin \text{relint } Q$,

the properties

$$\forall x_1 \in \gamma : x \prec x_1 \Rightarrow |x - y| \leq |x_1 - y| \quad (44)$$

hold.

Remark 6.4. Let \mathfrak{F} be a convex stratification; let (γ, \mathfrak{G}) be an EC with $\mathfrak{F} \subset \mathfrak{G}$. Then (γ, \mathfrak{F}) is an EC too.

Theorem 6.5. If $x(\cdot)$ is a viable steepest descent curve for a connected quasi convex family $\mathfrak{G} = \{\Omega_t\}_{t \in T}$ and $t \rightarrow x(t)$ is absolutely continuous then, $(x(\cdot), \mathfrak{G})$ is an expanding couple; however there exists a viable steepest descent curve γ for a connected quasi convex family $\mathfrak{G} = \{\Omega_t\}_{t \in T}$, with (γ, \mathfrak{G}) not expanding couple.

Proof. Let us observe that since $x(\cdot)$ is absolutely continuous then $t \rightarrow |x(t) - y|^2$ is not decreasing if and only if

$$\langle \dot{x}(t), x(t) - y \rangle \geq 0, \quad \forall y \in \Omega_{t'}, \text{ a.e. } t \geq t'.$$

The previous inequality is equivalent to the differential inclusion (43) of definition 6.1. Thus $(x(\cdot), \mathfrak{G})$ is an expanding couple.

To construct an example of viable steepest descent curve γ associated to \mathfrak{G} such that (γ, \mathfrak{G}) is not an expanding couple, let us consider the Cantor function $g : [0, 1] \rightarrow [0, 1]$, see [13, p. 83]. Let γ be the graph of g in the x_1, x_2 coordinate plane. The parametrization of $\gamma : x(t) = (x_1(t) = t, x_2(t) = g(t)), t \in [0, 1]$ is not absolutely continuous. Let

$$\Omega_t = \text{co}(\{(x_1, x_2) : 0 \leq x_1 \leq t, g(t) \leq x_2 \leq 1\}) \quad \text{for } t \in [0, 1].$$

As the Hausdorff distance between $\Omega_{t_1}, \Omega_{t_2}$ is $|t_1 - t_2|$, then $\{\Omega_t\}_{t \in T}$ is connected. $\dot{x}(t) = (1, 0)$ exists a.e in $[0, 1]$ and $\dot{x}(t) \in N_{\Omega_t}(x(t))$ since the halfplane $\{x_1 \geq t\}$ supports Ω_t at $x(t)$. Of course $x(t) = (t, g(t)) \in \partial \Omega_t$, so γ is a viable steepest descent curve for $\{\Omega_t\}_{t \in [0,1]}$. Let us notice now that

$$x(2/3) = (2/3, 1/2) \notin \text{relint } \Omega_{2/3},$$

and let us consider $x(1) = (1, 1), y \equiv (2/3, 1) \in \Omega_{2/3}$. Then (γ, \mathfrak{E}) is not an EC since

$$|x(2/3) - y| = 1/2 > |x(1) - y| = 1/3$$

holds. \square

To construct an expanding couple with the related curve which is not a viable steepest descent curve, let us consider the family $\{\Omega_t\}_{t \in [0,2]} = \{D_t\}_{t \in [0,1]} \cup \{E_t\}_{t \in (1,2]}$ of the example 6.2. Let $x(\cdot)$ be the continuous path with end point $\bar{x} = (\bar{x}_1, \bar{x}_2, 1) \in \partial E_2 \cap \{x_3 = 1\} \cap \{0 < x_1^2 + x_2^2 < 1\}$ defined as follows:

$$x(t) = \begin{cases} (\bar{x}_1, \bar{x}_2, t) & t \in (1, 2] \\ (\bar{x}_1, \bar{x}_2, 1) & t \in (\tau, 1] \text{ with } \tau = \bar{x}_1^2 + \bar{x}_2^2 > 0 \\ \frac{t}{\tau} \cdot x(\tau) & t \in [0, \tau]. \end{cases} \quad (45)$$

It is not difficult to see that the constructed curve together with the family of example 6.2 is an EC but $x(\cdot)$ is not a viable steepest descent curve.

Theorem 6.6. *Let \mathfrak{F} be a convex stratification and let γ be a self expanding path enclosed in $\max \mathfrak{F}$. Assume that (γ, \mathfrak{F}) satisfies (i) and (ii) of the definition 6.3. The following two facts are equivalent.*

- (i) *The couple (γ, \mathfrak{F}) is an expanding couple (EC);*
- (ii) *$\forall Q \in \mathfrak{F}, \forall x' \in \gamma \cap \partial_{rel} Q, \forall \gamma_1 \subset Q, \gamma_1$ a SEP with endpoint x' , then the set $\gamma_2 := \gamma_1 \cup (\gamma \setminus \gamma_{x'})$ (linearly ordered starting with the first point of γ_1) is a SEP.*

Proof. (i) \Rightarrow (ii)

Let $x_1, x_2, x_3 \in \gamma_2$ with $x_1 \prec x_2 \prec x_3$. If $x_1, x_2, x_3 \in \gamma_1$ or $x_1, x_2, x_3 \in \gamma \setminus \gamma_{x'}$, then (22) holds. If $x_1 \in \gamma_1 \subset Q$ and $x_2, x_3 \in \gamma \setminus \gamma_{x'}$, then (22) follows from (44). If $x_1, x_2 \in \gamma_1$ and $x_3 \in \gamma \setminus \gamma_{x'}$, then $|x_1 - x_2| \leq |x_1 - x'|$; as $x_1 \in Q$, by (44) $|x_1 - x'| \leq |x_1 - x_3|$; the last two inequalities imply (22). So γ_2 is a SEP.

(ii) \Rightarrow (i)

Let $Q \in \mathfrak{F}, y \in Q, \max(\gamma \cap \partial_{rel} Q) = x'$. Let γ_1 be the segment yx' and $\gamma_2 := \gamma_1 \cup (\gamma \setminus \gamma_{x'})$: as γ_2 is a SEP then (44) holds. \square

Theorem 6.7. *Let $K_1 \subset K_2$ be convex bodies in \mathbb{R}^n and let \mathfrak{F} be a convex stratification with minimum and maximum sets K_1, K_2 respectively. Let (γ, \mathfrak{F}) be an expanding couple, then*

- (i) *let $c_n^{(1)}$ be the constant in theorem 4.10; then*

$$(2c_n^{(1)})^{-1} \|\gamma \setminus K_1\| \leq \text{dist}(K_1, K_2); \quad (46)$$

- (ii) *there exists a constant c depending on the diameter of K_2 such that the bound*

$$\|\gamma \setminus K_1\| \leq c(w(K_2) - w(K_1))^{1/p} \quad (47)$$

holds for $p = n$;

- (iii) *when $n > 1$, for any $p \geq 1$, does not exist a constant c , not depending on K_2 , for which (47) holds.*

Proof. Let x be the last point of γ and x' be the projection of x onto K_1 , then

$$|x - x'| \leq \text{dist}(K_1, K_2). \quad (48)$$

The SEP γ , by theorem 6.6, can be extended and made it starting in $x' \in K_1$. Then

$$\gamma \supseteq \gamma \setminus K_1;$$

by the monotonicity property of γ

$$\text{co}(\gamma) \subseteq B(x', |x - x'|).$$

Then $\|\gamma \setminus K_1\| \leq \|\gamma\|$ and by (28), (48)

$$\|\gamma\| \leq c_n^{(1)} w(\text{co}(\gamma)) \leq c_n^{(1)} w(B(x', |x - x'|)) \leq 2c_n^{(1)} \text{dist}(K_1, K_2).$$

This proves (i). Inequality (47) follows immediately from (i) and inequality (19).

Let us observe now that, for any couple of nested convex bodies K_1, K_2 , the segment joining the points $x_i \in K_i$, for $i = 1, 2$ such that $|x_2 - x_1| = \text{dist}(K_2, K_1)$ is a special SEP which together with the trivial convex stratification $\{K_i\}_{i=1,2}$ is an expanding couple. So in order to prove (iii) it is enough to show that there exists a sequence of couples of nested convex bodies $\emptyset \neq K_{1,\nu} \subset K_{2,\nu}$ of \mathbb{R}^n satisfying

$$\frac{\text{dist}(K_{1,\nu}, K_{2,\nu})}{(w(K_{2,\nu}) - w(K_{1,\nu}))^{1/p}} \rightarrow \infty \quad \text{as } \nu \rightarrow \infty.$$

The example will be given in \mathbb{R}^2 ; however, it could be easily adapted to \mathbb{R}^n . Let $K_{1,\nu}$ be a family of segments of length α_ν/ν , where α_ν is a suitable positive real sequence to be determined in the sequel. Let us choose on the axis of $K_{1,\nu}$ a point p_ν of distance $1/\nu$ from $K_{1,\nu}$. Let

$$K_{2,\nu} = \text{co}(K_{1,\nu} \cup \{p_\nu\}).$$

Then

$$\text{dist}(K_{1,\nu}, K_{2,\nu}) = \frac{1}{\nu}, \quad w(K_{2,\nu}) - w(K_{1,\nu}) = \frac{1}{\pi\nu}(\sqrt{4 + \alpha_\nu^2} - \alpha_\nu).$$

The ratio

$$\frac{\text{dist}(K_{1,\nu}, K_{2,\nu})}{(w(K_{2,\nu}) - w(K_{1,\nu}))^{1/p}} = \nu^{\frac{1}{p}-1} \pi^{1/p} (\sqrt{4 + \alpha_\nu^2} - \alpha_\nu)^{-1/p}$$

is unbounded for $\nu \rightarrow \infty$ as $\alpha_\nu = \nu^q$, with $q > 1$, $q/p > 1 - 1/p$. \square

Lemma 6.8. *Let (γ, \mathfrak{F}) be an expanding couple and let $Q \in \mathfrak{F}$. Then $\gamma \cap \partial_{\text{rel}} Q$ is at most one point.*

Proof. The example 6.2 shows that $\gamma \cap \partial_{\text{rel}} Q$ may be empty. Assume that there are $x, x_1 \in \gamma \cap \partial_{\text{rel}} Q$, $x \prec x_1$. Choose $y = \frac{x+2x_1}{3} \in Q$; then (44) does not hold. \square

The following is a theorem of completeness of EC.

Theorem 6.9. *Let (γ, \mathfrak{F}) be an expanding couple and let $\Sigma = [w(\min \mathfrak{F}), w(\max \mathfrak{F})]$; then, there exists a connected quasi convex family (parameterized with respect to the mean width) $\mathfrak{G} = \{\Omega_t\}_{t \in \Sigma}$, containing the family \mathfrak{F} , and a continuous parametrization of γ : $\Sigma \ni t \rightarrow x(t) \in \gamma$, with the properties:*

- (i) the couple (γ, \mathfrak{G}) is an expanding couple with $\min \mathfrak{F} = \min \mathfrak{G}$, $\max \mathfrak{F} = \max \mathfrak{G}$;
- (ii) $x(\cdot)$ is a continuous map from $\Sigma \rightarrow \mathfrak{G}$;
- (iii) for all $t \in \Sigma$ the point $x(t) \in \Omega_t$; moreover $t' < t'', x(t') \neq x(t'')$ imply $x(t'') \notin \Omega_{t'}$;
- (iv) $\forall t' \in [\min \Sigma, \max \Sigma], \forall y \in \Omega_{t'}$, the real function $t \rightarrow |x(t) - y|^2$ is not decreasing for $t \in (t', \max \Sigma]$.

Proof. Let us parameterize the elements Q of \mathfrak{F} by their mean widths. The family \mathfrak{F} can be augmented to a convex stratification \mathfrak{F}_1 , adding the sets

$$Q_s := \lim_{\tau \rightarrow s, \tau \in w(\mathfrak{F})} Q_\tau,$$

if they are not present. Then, the couple (γ, \mathfrak{F}_1) is still an expanding couple and $w(\mathfrak{F}_1)$ is closed. If \mathfrak{F}_1 it is not a connected quasi convex family, let us augment it in the following way. Let us consider the mean width function w mapping \mathfrak{F}_1 to a subset $w(\mathfrak{F}_1)$ of Σ . Let $\tau \in \Sigma \setminus w(\mathfrak{F}_1)$. Let (τ_1, τ_2) the maximal interval enclosed in $\Sigma \setminus w(\mathfrak{F}_1)$ containing τ . Then let us consider the annulus between the convex sets $Q_{\tau_1}, Q_{\tau_2} \in \mathfrak{F}_1$. Let us assume that

$$\gamma_{\tau_1, \tau_2} := \gamma \cap \text{relint}(Q_{\tau_2} \setminus Q_{\tau_1}) \neq \emptyset;$$

then γ_{τ_1, τ_2} is not a single point; let us complete the stratification \mathfrak{F}_1 between Q_{τ_1}, Q_{τ_2} in the following way: for any $x \in \gamma_{\tau_1, \tau_2}$ let us add to \mathfrak{F}_1 the set $\text{co}(Q_{\tau_1} \cup \gamma_x)$. Let \mathfrak{F}_2 the augmented family so obtained. Of course \mathfrak{F}_2 contains \mathfrak{F}_1 ; moreover by construction (γ, \mathfrak{F}_2) is an EC, $w(\mathfrak{F}_2)$ is closed and $w(\mathfrak{F}_2) \supset w(\mathfrak{F}_1)$. If $\Sigma \setminus w(\mathfrak{F}_2)$ is not empty, let $\tau \in \Sigma \setminus w(\mathfrak{F}_2)$; let (τ_1, τ_2) the maximal interval enclosed in $\Sigma \setminus w(\mathfrak{F}_2)$ containing τ . The way as \mathfrak{F}_2 has been constructed implies that $\gamma \cap \text{relint } Q_{\tau_2} \setminus \text{relint } Q_{\tau_1} = \emptyset$, but $\gamma \cap \partial_{\text{rel}} Q_{\tau_2} \cap \partial_{\text{rel}} Q_{\tau_1} \neq \emptyset$. At has been noticed above, $\gamma \cap \partial_{\text{rel}} Q_{\tau_2} \cap \partial_{\text{rel}} Q_{\tau_1}$ consists in just one point. Let us complete \mathfrak{F}_2 in a connected quasi convex family (interpolating the couples Q_{τ_1}, Q_{τ_2} as in (42)). Let us call \mathfrak{G} the augmented family; then (γ, \mathfrak{G}) is an EC, $w(\mathfrak{G}) = \Sigma$ and \mathfrak{G} is connected.

Now let us parameterize γ . Let $x \in \gamma$. Let

$$t^-(x) = \sup\{w(Q) : Q \in \mathfrak{G}, x \notin Q\}, t^+(x) = \min\{w(Q) : Q \in \mathfrak{G}, x \in Q\}.$$

Let $I(x) = [t^-(x), t^+(x)]$. The set valued map $x \rightarrow I(x)$ from γ to the metric space of closed subintervals of Σ is strictly "monotone", to say

$$x_1 \prec x_2 \implies \max I(x_1) < \min I(x_2). \quad (49)$$

This is a consequence of the following two properties:

- (I) the subclass of sets of $Q \in \mathfrak{G}, x_1 \in Q$ is contained in those containing x_2 ;
- (II) does not exist an element Q of \mathfrak{G} , such that $\partial_{\text{rel}} Q$ contains the arc $\widehat{x_1 x_2}$ of γ .

Let $t \in \Sigma$, then $t \in I(x)$ for a suitable $x \in \gamma$. This gives us a continuous parametrization $t \rightarrow x(t)$ of γ . Then (ii) and the first sentence of (iii) are proved. Now let $t' < t'', x(t') \neq x(t'')$; this implies $x(t') \prec x(t'')$; then (49) gives us $x(t'') \notin \Omega_{t'}$. Property (iv) follows from (44), since as observed above, (γ, \mathfrak{G}) is an EC. \square

Definition 6.10. Let (γ, \mathfrak{G}) be an EC with $\mathfrak{G} = \{\Omega_t\}_{t \in T}$ a parameterized connected quasi convex family. Let

$$x(t) = \max\{x \in \gamma : x \in \Omega_t\}. \quad (50)$$

The map $t \rightarrow (x(t), \Omega_t)$ will be called a **(joint) parametrization** of the EC (γ, \mathfrak{G}) .

Let (γ, \mathfrak{G}) be an EC with \mathfrak{G} connected. Let us point out that:

- (a) as in the previous theorem and definition a continuous parametrization t for \mathfrak{G} can be used to give a parametrization of γ . However $t \rightarrow x(t) \in \gamma$ may have sets of constancy. This occurs as example in the case a point $x \in \gamma$ belongs simultaneously to the boundary of all sets Ω_t for $t_1 \leq t \leq t_2$;
- (b) a 1-1 parametrization in curvilinear abscissa of γ could be used to make a parametrization of \mathfrak{G} : to a $Q \in \mathfrak{G}$ it is associated s such that $x(s) \in \partial_{rel} Q$; but some sets of \mathfrak{G} may be lost. See the curve γ defined by (45) associated to the family in the example 6.2 where a such parametrization for \mathfrak{G} would have jumps.

Lemma 6.11. Let (γ, \mathfrak{G}) be an EC, with \mathfrak{G} a connected quasi convex family and if

$$\text{for every } Q \in \mathfrak{G}, Q \neq \min \mathfrak{G}, \text{ the dimension of } \text{Aff}(Q) \text{ is constant} \quad (51)$$

then there exists a joint parametrization $t \rightarrow (x(t), \Omega_t)$ of (γ, \mathfrak{G}) such that $t \rightarrow x(t)$ is a viable steepest descent curve for $\{\Omega_t\}$.

Proof. Let us assume with no loss of generality that the dimension of elements Q is n . Let us choose the curvilinear abscissa of γ as the parameter t . Let

$$\Omega_t := \min\{Q \in \mathfrak{G} : x(t) \in Q, \quad 0 \leq t \leq \|\gamma\|\}.$$

The fact that \mathfrak{G} is connected and (51) imply that $x(t) \in \partial \Omega_t$. The proof of (ii) of the definition 6.1 can be done as in the proof of theorem 6.5. \square

Remark 6.12. Let $(x(t), \Omega_t)_{t \in T}$ be a joint parametrization of an EC with $\{\Omega_t\}_{t \in T}$ a connected quasi convex family and let $x(t_0) \in \text{relint } \Omega_{t_0}$. Then there exists an interval, not reduced to a point, containing t_0 where $x(t) = x(t_0)$.

The following problem is addressed and solved in what follows.

Given a connected quasi convex family $\mathfrak{G} = \{\Omega_t\}_{t \in \Sigma}$ and a point $\bar{x} \in \partial \max \mathfrak{G}$; does it exist a curve γ with end point \bar{x} so that the couple (γ, \mathfrak{G}) is a EC ? Is it γ unique ?

Definition 6.13. Let $(\gamma^{(m)}, \mathfrak{G}^{(m)})$ be a sequence of EC with $\mathfrak{G}^{(m)}$ connected quasi convex families and let $\min \mathfrak{G}^{(m)} = \Omega_0$, $\max \mathfrak{G}^{(m)} = \Omega$. Let $s^{(m)} \in [0, l^{(m)}]$ be the arc length of $\gamma^{(m)}$, and let us choose for all of them the same parameter $t \in [0, 1]$, $t = s^{(m)} / l^{(m)}$. Let us define

$$\lim_{m \rightarrow \infty} (\gamma^{(m)}, \mathfrak{G}^{(m)}) = (\gamma, \mathfrak{G})$$

if the sequence $[0, 1] \ni t \rightarrow x^{(m)}(t)$ uniformly converges to $t \rightarrow x(t)$ of γ and $\mathfrak{G}^{(m)}$ converges to \mathfrak{G} according to definition 5.9.

Theorem 6.14. *Let $(\gamma^{(m)}, \mathfrak{G}^{(m)})$ be a sequence of EC with $\mathfrak{G}^{(m)}$ connected quasi convex families and let $\min \mathfrak{G}^{(m)} = \Omega_0$, $\max \mathfrak{G}^{(m)} = \Omega$. Then there exists a subsequence which converges to an EC (γ, \mathfrak{G}) .*

Proof. Let us parameterize all $\mathfrak{G}^{(m)}$ by the mean width $w \in W = [w(\Omega_0), w(\Omega)]$, i.e. $\mathfrak{G}^{(m)} = \{\Omega_w^{(m)}\}_{w \in W}$. Then, each $\mathfrak{G}^{(m)}$ is represented as a continuous function defined in W and valued into the metric space \mathfrak{K} of all compact subsets of Ω with Hausdorff distance. From proposition 3.3 these functions are equicontinuous; from Ascoli-Arzelá theorem [14, p. 234] there exists a converging subsequence to a continuous function, corresponding to a family $\mathfrak{G} = \{\Omega_w\}_{w \in W}$. Let us assume for simplicity that the converging subsequence is the initial sequence. Let us consider now the sequence of the corresponding curves $\gamma^{(m)}$. Let $l^m = \|\gamma^{(m)}\|$. Theorem 4.10 implies

$$0 < l^m < c_n^{(1)} w(\Omega).$$

Let us reparameterize $\gamma^{(m)}$ with their arc length $s^{(m)} \in [0, l^m]$, and let us choose for all of them the same parameter $t \in [0, 1]$, $t = s^{(m)}/l^m$. From theorem 4.10 they are equilipschitz, then there exists a subsequence converging to a self expanding path $\gamma = \{x(t), t \in [0, 1]\}$. It remains to prove that (γ, \mathfrak{G}) is a EC.

(i) and (ii) of definition 6.3 hold with a limit argument.

To prove (iii), let $Q \in \mathfrak{G}$, $x \prec x' \prec x_1 \in \gamma$, $x \notin \text{relint } Q$, $x' = x(t')$, $x_1 = x(t_1)$. The previous argument implies that $Q = \lim_{m \rightarrow \infty} \Omega_w^{(m)}$, with $w = w(Q)$, $x = x(t) = \lim_{m \rightarrow \infty} x^{(m)}(t)$, $x' = x(t') = \lim_{m \rightarrow \infty} x^{(m)}(t')$, $x_1 = x(t_1) = \lim_{m \rightarrow \infty} x^{(m)}(t_1)$. Since Q is compact, then eventually $x^{(m)}(t') \notin \Omega_w^{(m)}$. Since $(x^{(m)}(\cdot), \mathfrak{G}^{(m)})$ are EC, they satisfy definition 6.3. It follows that for m large enough

$$\forall y \in \Omega_w^{(m)} \implies |x^{(m)}(t') - y| \leq |x^{(m)}(t_1) - y|.$$

Then, for every $y \in Q$ it holds $|x' - y| \leq |x_1 - y|$. Thus, as x' tends to x , (44) holds. \square

The following is an existence result for expanding couples.

Theorem 6.15. *Let $\mathfrak{G} = \{\Omega_t\}_{t \in T}$ be a connected quasi convex family in \mathbb{R}^n with $t = w(\Omega_t)$, $T = [w(\min \mathfrak{G}), w(\max \mathfrak{G})]$ then*

- (i) *for every $\bar{x} \in \partial \max \mathfrak{G}$ there exists a self expanding path γ , so that (γ, \mathfrak{G}) is an EC with γ having end point \bar{x} ;*
- (ii) *γ is rectifiable, with length bounded by the mean width of \mathfrak{G} times a constant depending only on n ;*
- (iii) *γ can be uniformly approximated with piecewise linear self expanding curves $x^{(m)}(\cdot)$, related to connected quasi convex families $\mathfrak{G}^{(m)}$ (uniformly converging to \mathfrak{G}) and $(x^{(m)}(\cdot), \mathfrak{G}^{(m)})$ are joint parameterized EC.*

Proof. Let $\bar{w} = w(\max \mathfrak{G})$, $w_0 = w(\min \mathfrak{G})$.

Let D be a countable dense subset of $[w_0, \bar{w}]$, and let $w_0 = w_1^{(m)} < \dots < w_m^{(m)} = \bar{w}$ elements of D , satisfying $\{w_1^{(m)}, \dots, w_m^{(m)}\} \subset \{w_1^{(m+1)}, \dots, w_{m+1}^{(m+1)}\}$. Let $\{\Omega_{w_1^{(m)}}, \dots, \Omega_{w_m^{(m)}}\}$ be a finite convex

stratification (extracted from \mathfrak{G}) and let $\{O_t^{(m)}\}_{t \in [w_0, \bar{w}]}$ the related connected family as defined in the proof of theorem 5.8. Let $t \rightarrow x^{(m)}(t)$ the piecewise linear curve ending at \bar{x} , union of segments joining the m points $p^{(1)}, \dots, p^{(m)}$, where $p^{(m)} = \bar{x}$, $p^{(j-1)}$ is the projection of $p^{(j)}$ onto $\Omega_{w_{j-1}^{(m)}}$, $j = 2, \dots, m$. Let

$$x^{(m)}(t) = \frac{(t - w_{j-1}^{(m)})p^{(j)} + (w_j^{(m)} - t)p^{(j-1)}}{w_j^{(m)} - w_{j-1}^{(m)}}, \quad w_{j-1}^{(m)} \leq t \leq w_j^{(m)}, \quad (52)$$

the corresponding point between $p^{(j-1)}$ and $p^{(j)}$. If $p^{(j-1)} \neq p^{(j)}$, the unit segment direction d_j of $p^{(j-1)}p^{(j)}$, is orthogonal at each point $x^{(m)}(t)$ of $p^{(j-1)}p^{(j)}$ to the boundary of convex set $O_t^{(m)}$ which is an interpolation between $\Omega_{w_{j-1}^{(m)}}$ and $\Omega_{w_j^{(m)}}$ as defined in (42). Therefore

$$\frac{dx^{(m)}}{dt}(t) \in N_{O_t^{(m)}}(x^{(m)}(t)) \quad \text{for} \quad w_{j-1}^{(m)} \leq t \leq w_j^{(m)}.$$

Thus for a small $\varepsilon > 0$, $p^{(j-1)}p^{(j)} \cap O_{w_{j-1}^{(m)} + \varepsilon}^{(m)}$ is the trajectory of a viable steepest descent curve of the quasi convex family

$$\{O_t^{(m)}\}_{t \in [w_{j-1}^{(m)} + \varepsilon, w_j^{(m)} - \varepsilon]}.$$

From theorem 6.5,

$$t \rightarrow (x^{(m)}(t), O_t^{(m)})_{t \in [w_{j-1}^{(m)}, w_j^{(m)}]}$$

is a joint parametrization of $(x^{(m)}(\cdot), \{O_t^{(m)}\}_{t \in [w_{j-1}^{(m)}, w_j^{(m)}]})$; of course $T \ni t \rightarrow x^{(m)}(t)$ is a SEP, and $(x^m(\cdot), \mathfrak{G}^{(m)})$ with $\mathfrak{G}^{(m)} = \{O_{t \in T}^{(m)}\}$ are expanding couples. By construction the sequence of families $\mathfrak{G}^{(m)}$ converges uniformly in the Hausdorff distance to \mathfrak{G} . From theorem 6.14 there exists a subsequence of $(x^m(\cdot), \mathfrak{G}^{(m)})$ converging to an expanding couple (γ, \mathfrak{G}) . Thus (i) and (iii) are proved and (ii) follows from (28). \square

From theorem 5.8, previous theorem and remark 6.4 it follows

Corollary 6.16. *Let \mathfrak{F} be a convex stratification and let $\bar{x} \in \partial \max \mathfrak{F}$. Then, there exists a SEP γ with end point \bar{x} so that (γ, \mathfrak{F}) is an EC.*

The following lemma and theorem show that given an EC (γ, \mathfrak{G}) with \mathfrak{G} a connected quasi convex family, a joint parametrization for (γ, \mathfrak{G}) that gives an absolutely continuous parametrization (indeed lipschitz) for γ can be found.

Lemma 6.17. *Let $s : [w_0, \underline{w}] \rightarrow \mathbb{R}$ be a continuous not decreasing function and let η be the plane curve $(w, s(w))$, $w_0 \leq w \leq \underline{w}$. Then there exists a continuous, strictly increasing function $\tau : [w_0, \underline{w}] \rightarrow [0, \|\eta\|]$, with inverse $w(\cdot)$, so that $s(w(\cdot)) : [0, \|\eta\|] \rightarrow \mathbb{R}$ is a lipschitz function.*

Proof. Let

$$\tau(w) := \|\eta \cap ([w_0, w] \times \mathbb{R})\|.$$

The function $w \rightarrow \tau(w)$ is a continuous, strictly increasing function (see e.g. [17, theorem 8.4]) with inverse

$$w : [0, \|\eta\|] \rightarrow [w_0, \underline{w}].$$

Let $0 \leq \tau_1 < \tau_2 \leq \|\eta\|$, then:

$$|s(w(\tau_2)) - s(w(\tau_1))| \leq \|\eta \cap ([w(\tau_1), w(\tau_2)] \times \mathbb{R})\| = \tau_2 - \tau_1.$$

This concludes the proof. \square

Theorem 6.18. *Let (γ, \mathfrak{G}) be an expanding couple with \mathfrak{G} a connected quasi convex family. Then there exists a joint parametrization $(z(\tau), \Omega_\tau)_{\tau \in T}$ of (γ, \mathfrak{G}) so that $t \rightarrow z(t)$ is lipschitz.*

Proof. Let us start from the results of theorem 6.9. If the mean width parameter w of elements of the family $\mathfrak{G} = \{O_w\}_{w \in [w(\min \mathfrak{G}), w(\max \mathfrak{G})]}$ is used to parameterize γ as in (50), a continuous monotone parametrization $x(w), w \in [w(\min \mathfrak{G}), w(\max \mathfrak{G})]$ of γ is obtained. From theorem 4.10 the curve γ is rectifiable and can be represented as function of its arc length: $y(s), 0 \leq s \leq \|\gamma\|$.

Let $s : [w(\min \mathfrak{G}), w(\max \mathfrak{G})] \rightarrow \mathbb{R}$ the map defined as

$$w \rightarrow s(w) = \|\gamma \cap O_w\| = \|\gamma_{x(w)}\|.$$

$s(\cdot)$ is a continuous not decreasing function and $y(s(w)) = x(w)$. From previous lemma there exists a continuous change of variable $w = w(\tau)$ so that $\tau \rightarrow s(w(\tau))$ is lipschitz continuous. The connected quasi convex family $\mathfrak{G} = \{O_w\}$ will be changed in $\Omega_\tau = O_{w(\tau)}$ with $\tau \in [0, \|\text{graph}(s)\|]$; the associated curve γ has the parametrization $z(\tau) = y(s(w(\tau)))$; since $s \rightarrow y(s)$ is lipschitz continuous, $z(\cdot) : \tau \rightarrow y(s(w(\tau)))$, composition of two lipschitz continuous maps, is lipschitz continuous. \square

Example 6.19. *Let $[0, 1] \ni t \rightarrow g(t)$ be the Cantor function. Let $\mathfrak{F} = \{\Omega_t\}$, with*

$$\Omega_t = \{x \in \mathbb{R}^2 : |x| \leq \frac{g(t) + t}{2}, 0 \leq t \leq 1\}$$

a connected quasi convex family of concentric circles, and let $x(\tau) = \frac{g(\tau) + \tau}{2} \bar{x}$ ($\bar{x} \in \partial\Omega_1$) be the parametrization of the radius from the origin to \bar{x} . It is a viable steepest descent curve for \mathfrak{F} , but its parametrization is not absolutely continuous.

However, let us notice that, using in place of t the mean width parameter w of the family $\{\Omega_t\}$, i.e. $w(\Omega_t) = g(t) + t$ and the parametrization of γ given by (50), the absolutely continuous property for the parametrization γ is restored. Next proposition shows that, given a connected quasi convex family $\{\Omega_t\}_{t \in T}$, viable steepest descent curves are continuously depending on their end point, if they are absolutely continuous.

Proposition 6.20. *Let $x(\cdot), y(\cdot)$ be two absolutely continuous viable steepest descent curves of a connected quasi convex family $\{\Omega_t\}_{t \in T}$ with end points \bar{x}, \bar{y} respectively. Then*

$$|x(t) - y(t)| \leq |\bar{x} - \bar{y}| \quad \forall t \in T, \tag{53}$$

and there is at most one absolutely continuous viable steepest descent curve with given end point.

Proof. Let t be a value for which $\dot{x}(t) \in N_{\Omega_t}(x(t))$, $\dot{y}(t) \in N_{\Omega_t}(y(t))$. Since $x(t), y(t) \in \partial_{rel}\Omega_t$, the previous inclusions mean that

$$\langle \dot{x}(t), x(t) - y(t) \rangle \geq 0, \quad \langle \dot{y}(t), y(t) - x(t) \rangle \geq 0. \quad (54)$$

Adding the two previous inequalities, a.e.

$$\frac{1}{2} \frac{d}{dt} |x(t) - y(t)|^2 \geq 0$$

holds. The absolute continuity assumption implies that the distance between $x(t)$ and $y(t)$ is not decreasing with respect to the level value t ; this proves (53). \square

Next theorem gives us a continuous dependence for EC .

Theorem 6.21. *Let $\mathfrak{G} = \{\Omega_t\}_{t \in T}$ be a connected quasi convex family and let $(x_1(\cdot), \mathfrak{G}), (x_2(\cdot), \mathfrak{G})$ be two expanding couples with joint continuous parametrizations; let $\overline{x}_1, \overline{x}_2$ be the curves' end points. Then for all $t \in T$*

$$|x_1(t) - x_2(t)| \leq |\overline{x}_1 - \overline{x}_2| \quad (55)$$

and for any given end point there exists at most one EC .

Proof. Step 1. Let $s_i(\cdot) : [w_0, \overline{w}] \rightarrow \mathbb{R}$ be continuous not decreasing functions ($i = 1, 2$). Then, there exists a real continuous, strictly increasing function $\tau : [w_0, \overline{w}] \rightarrow \tau([w_0, \overline{w}]) := T_1$, with inverse map $w(\cdot)$, so that $s_i(w(\cdot)) : T_1 \rightarrow \mathbb{R}$ is a lipschitz function ($i = 1, 2$).

The proof of step 1 is similar to the proof of lemma 6.17 where η is the three dimensional curve $[w_0, \overline{w}] \ni w \rightarrow (w, s_1(w), s_2(w))$ and

$$\tau(w) := \|\eta \cap ([w_0, w] \times \mathbb{R}^2)\|.$$

Step 2. Let (γ_i, \mathfrak{G}) be two expanding couples ($i = 1, 2$). Then, there exists a joint parametrization $(z_i(\tau), \Omega_\tau)_{\tau \in T_1}$ of (γ_i, \mathfrak{G}) ($i = 1, 2$) so that $z_i(\cdot)$ is lipschitz continuous in T_1 ($i = 1, 2$). The proof of step 2 is similar to the proof of theorem 6.18 where

$$w \rightarrow s_i(w) = \|\gamma_i \cap O_w\|, \quad i = 1, 2$$

and $w(\cdot)$ is introduced in step 1.

The conclusion of the proof of the theorem is similar to that of the proposition 6.20. Let us use the joint parametrizations $(z_1(\tau), \Omega_\tau), (z_2(\tau), \Omega_\tau)$ introduced in the previous step. Let us recall that $\tau \rightarrow z_1(\tau), \tau \rightarrow z_2(\tau)$ are absolutely continuous in T_1 . If $z_1(\tau), z_2(\tau) \in \partial_{rel}\Omega_\tau$ then (54) holds for such τ . In case $z_1(\tau_0) \in \text{relint } \Omega_{\tau_0}$ then, from remark 6.12, the right or the left derivative of $z_1(\cdot)$ at τ_0 is zero. Similarly is for $z_2(\cdot)$. Then, there exists a.e. $\dot{z}_1(\tau), \dot{z}_2(\tau)$ and (54) holds a.e. Thus a.e.

$$\frac{1}{2} \frac{d}{d\tau} |z_1(\tau) - z_2(\tau)|^2 \geq 0.$$

Therefore, by the absolute continuity property,

$$|z_1(\tau) - z_2(\tau)| \leq |\overline{x}_1 - \overline{x}_2|.$$

Then (55) holds for every joint parametrization of $(\gamma_1, \mathfrak{G}), (\gamma_2, \mathfrak{G})$. \square

Corollary 6.22. Let (γ_i, \mathfrak{G}) be two EC with \mathfrak{G} a connected quasi convex family ($i = 1, 2$). Let $x_i = \gamma_i \cap \max \mathfrak{G}$ ($i = 1, 2$). Then

$$\text{dist}(\gamma_1, \gamma_2) \leq |x_1 - x_2|.$$

Proof. By definition of EC,

$$|x(t_1) - y(\tau)| \leq |x(t_1) - y(t_1)|, \quad \text{for } \tau < t_1,$$

since $y(\tau) \in \Omega_\tau \subset \Omega_{t_1}$. \square

7 Appendix

Lemma 7.1. Let $K_{v,\delta}$ be a circular n -dimensional cone of axis v , amplitude δ greater than zero and less than $\pi/2$. Then

$$\int_{\widehat{K_{v,\delta}}} \langle \theta, v \rangle d\sigma(\theta) = \frac{\omega_{n-1}}{n-1} \sin^{n-1} \delta.$$

Proof. See e.g. [15, pp. 223-224]. \square

Lemma 7.2. Let $K_{v,\alpha/2}$ be a circular n -dimensional cone of axis v , amplitude $\frac{\alpha}{2}$ greater than zero and less than $\pi/2$. Let u a unit vector in $K_{v,\alpha/2}^*$, then

$$\int_{\widehat{K_{v,\alpha/2}}} \langle \theta, u \rangle d\sigma(\theta) \geq \frac{\omega_{n-1}}{n-1} \sin^n(\alpha/4).$$

Proof. Let $v = e_n$. Let $\phi = \langle \theta, e_n \rangle, \psi = \langle u, e_n \rangle$. For θ in the sector $\widehat{K_{v,\alpha/2}}$, $0 \leq \phi \leq \alpha/2$; and $0 \leq \psi \leq \frac{\pi}{2} - \alpha/2$ since $u \in K_{e_n,\alpha/2}^*$. So $0 \leq \phi + \psi \leq \pi/2$. Moreover the distance on the sphere S^{n-1} between θ and u is less than the sum of the distances between θ and v , v and u , i.e

$$\langle \theta, u \rangle \geq \cos(\phi + \psi).$$

Let us consider the smallest sector $\widehat{K_{v,\alpha/4}}$; the inequality

$$\int_{\widehat{K_{v,\alpha/2}}} \langle \theta, u \rangle d\sigma(\theta) \geq \int_{\widehat{K_{v,\alpha/4}}} \cos(\phi + \psi) d\sigma(\theta)$$

holds. For $\theta \in \widehat{K_{v,\alpha/4}}$, the angle $0 \leq \phi \leq \alpha/4$, therefore $0 \leq \phi + \psi \leq \frac{\pi}{2} - \alpha/4$. Thus

$$\cos(\phi + \psi) \geq \cos(\pi/2 - \alpha/4) = \sin(\alpha/4).$$

Then

$$\int_{\widehat{K_{e_n,\alpha/4}}} \cos(\phi + \psi) d\sigma(\theta) \geq \sin(\alpha/4) \int_{\widehat{K_{e_n,\alpha/4}}} \cos \phi d\sigma(\theta).$$

From previous lemma the proof is obtained. \square

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